**Problem 1**

1. Lemma: cos(tan-1()) =

Proof: In right triangle ABC, ∠B = 90. Let AB = x, BC = 1, and = .

According to the Pythagorean theorem, AC = = .

tan-1() = , cos(tan-1()) = cos( = .

Let an = cos(tan-1(an-1)), since

Therefore, an = cos(tan-1(an-1)) = , which is the recursion formula we want to find.

1. Since the problem assumes the numbers converge, as n approaches infinity, .

Thus, an = = ,

By solving ) = 1,

We get = .

**Problem 2**

To find the average scores Susan and Eric are each going to get for one turn, we can use expectation to calculate the weighted average for each of them.

Let P(i), where i, be the probability for each of the number on the die.

For Susan:

For every die she rolls, P(1) = P(2) = P(3) = … = P(6) = 1/6.

Case 1: Susan get a number other than 1.

Then the expectation for all the numbers other than 1 is

= 1/6\*2+1/6\*3+1/6\*4+…+1/6\*6 = 1/6\*(2+3+4+5+6) = 10/3.

Case 2: Susan get 1, and she needs to roll two new dice.

Let the two numbers she gets be a pair of (a, b),

For both a and b, P(1) = P(2) = P(3) = … = P(6) = 1/6, which is the same as case 1.

Therefore, for either a or b, the expectation for getting a number other than 1 is 1/6\*, and since there are two numbers a and b, = 2, If either a or b or both are 1, we go back to case 1 and repeat the process.

Therefore, the expectation for all possibilities for Susan can be obtained by

E1 + 2\*(1/6\* E1) + 4\*(1/36\* E1) + … + 2n\*[(1/6)^n\* E1]

= E1 + n \* E1

=

= 5

Therefore, Susan’s average score is 5.

For Eric:

For every die he rolls, P(even)= P(odd)=

Case 1: Eric get an odd total.

The expectation for two numbers should be

Lemma 1: =

Proof: let S = and S’ = 3S.

Then S’ = 3S = .

S’ – S = 1 - = 2S

Therefore, S = .